any such invariant family inside the γ-cones is contained in the unique invariant family E_m^+ of continuous plane fields obtained there. Since every tangent set $\tau_p\varphi_m^+$ projects onto $\mathbb{R}^k$, we conclude that $\tau_p\varphi_m^+ = (E_p^+)_m$, that is, the φ_m^+ are C^1 functions.

Smoothness of φ_m^- is proved likewise. This ends the proof of (i).

It remains to prove (iii). We remarked after the formulation of the theorem that we can construct the manifolds $(W_m^-)_p$ and $(W_m^+)_p$ for any point $p = (x, y)$. We still have $(W_m^+)_p = \text{graph}(\varphi_m^+)_p$ and $(W_m^-)_p = \text{graph}(\varphi_m^-)_p$ for some γ-Lipschitz functions $(\varphi_m^+)_p : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and $(\varphi_m^-)_p : \mathbb{R}^{n-k} \to \mathbb{R}^k$ and properties analogous to (i) and (ii).

Lemma 6.2.19. *For $p, q \in \mathbb{R}^n$ the intersection $(W_m^+)_p \cap (W_m^-)_q$ consists of exactly one point.*

Proof. If $z = (x, y) \in (W_m^+)_p \cap (W_m^-)_q$ then $x = (\varphi_m^-)_q(y)$ and $y = (\varphi_m^+)_p(x)$ and hence $x = (\varphi_m^-)_q \circ (\varphi_m^+)_p(x)$. This in turn implies again that $(x, (\varphi_m^+)_p(x)) \in (W_m^+)_p \cap (W_m^-)_q$. But since we can assume $\gamma < 1$ the map $(\varphi_m^-)_q \circ (\varphi_m^+)_p : \mathbb{R}^k \to \mathbb{R}^k$ is a contraction and hence has a unique fixed point. □

Now assume $p \notin (W_m^-)_0$. By Lemma 6.2.19 there is a unique $q \in (W_m^-)_0 \cap (W_m^+)_p$. Using (ii) for $(W_m^-)_0$ and $(W_m^+)_p$ we see that

$$\begin{aligned}
&\|f_{m+L-1} \circ \cdots \circ f_m(p)\| \\
&\quad \geq \|f_{m+L-1} \circ \cdots \circ f_m(p) - f_{m+L-1} \circ \cdots \circ f_m(q)\| - \|f_{m+L-1} \circ \cdots \circ f_m(q)\| \\
&\quad \geq (\mu')^L \|p - q\| - (\lambda')^L \|q\| = (\mu')^L \Big(\|p - q\| - \Big(\frac{\lambda'}{\mu'}\Big)^L \|q\| \Big).
\end{aligned}$$

Whenever $\lambda' < \nu < \mu'$ and $C \in \mathbb{R}$ this quantity will exceed $C \cdot \nu^L \|p\|$ for sufficiently large $L \in \mathbb{N}$.

Together with a parallel argument for $(W_m^+)_0$ this proves (iii) and thus also the uniqueness of W_m^+ and W_m^-.

This finishes the proof of the general part of the Hadamard–Perron Theorem.

Step 5. To complete the proof of Theorem 6.2.8 we now prove that in the hyperbolic case the leaves are as smooth as the diffeomorphism. We will, in fact, prove the stronger statement that if $\mu \geq 1$ in Theorem 6.2.8 then $\{W_m^+\}_{m \in \mathbb{Z}}$ consists of manifolds as smooth as the diffeomorphism. Df_m has block form $\begin{pmatrix} A_m^{uu} & A_m^{su} \\ A_m^{us} & A_m^{ss} \end{pmatrix}$ with A_m^{uu} a $k \times k$-matrix with $\|(A_m^{uu})^{-1}\| \leq 1/(\mu - \delta)$, A_m^{ss} an $(n-k) \times (n-k)$-matrix with $\|A_m^{ss}\| \leq \lambda + \delta$, and $\|A_m^{su}\| < \delta$, $\|A_m^{us}\| < \delta$. By the preceding steps, notably Lemma 6.2.16, we can obtain W_m^+ by taking smooth functions $\varphi_m^0 \in C_\gamma^0(\mathbb{R}^k)$ (such as $\varphi_m^0 = 0$), applying the graph transform repeatedly to obtain families $\{\varphi_m^i\}$ for $i \in \mathbb{N}$, and taking the limit as $i \to \infty$. We plan to show inductively that the $r + 1$st derivative of φ_m^i converges as $i \to \infty$, so long as f is C^{r+1}. To that end we note that $D\varphi_m^i$ is the graph of a linear map E_m^i from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$, or, equivalently, the image of the map

$\begin{pmatrix} I \\ E^i_m \end{pmatrix}: \mathbb{R}^k \to \mathbb{R}^n$. Notice that the image of $D\varphi^i_m$ under Df_m is the image of the linear map

$$\begin{pmatrix} A^{uu}_m & A^{su}_m \\ A^{us}_m & A^{ss}_m \end{pmatrix} \begin{pmatrix} I \\ E^i_m \end{pmatrix} = \begin{pmatrix} A^{uu}_m + A^{su}_m E^i_m \\ A^{us}_m + A^{ss}_m E^i_m \end{pmatrix}.$$

If, referring to (6.2.6), we let $g^i_m := (G^{m-1}_{\varphi^i_{m-1}})^{-1}$ then this has to coincide with the image of $\begin{pmatrix} I \\ E^{i+1}_{m+1} \circ (g^i_{m+1})^{-1} \end{pmatrix}$ which is the same as that of

$$\begin{pmatrix} A^{uu}_m + A^{su}_m E^i_m \\ (E^{i+1}_{m+1} \circ (g^i_{m+1})^{-1})(A^{uu}_m + A^{su}_m E^i_m) \end{pmatrix},$$

so

$$(E^{i+1}_{m+1} \circ (g^i_{m+1})^{-1})(A^{uu}_m + A^{su}_m E^i_m) = A^{us}_m + A^{ss}_m E^i_m.$$

Composing with g^i_{m+1} and differentiating r times we get

$$\begin{aligned} D^r E^{i+1}_{m+1}(\alpha^u_{m+1,i+1})^{-1} &+ E^{i+1}_{m+1}(A^{su}_m \circ g^i_{m+1})(D^r E^i_m \circ g^i_{m+1})(Dg^i_{m+1})^{\otimes r} \\ &= (A^{ss}_m \circ g^i_{m+1})(D^r E^i_m \circ g^i_{m+1})(Dg^i_{m+1})^{\otimes r} + \zeta_{m+1,i+1}(\alpha^u_{m+1,i+1})^{-1}, \end{aligned}$$

where $\zeta_{m+1,i+1}$ is a polynomial in lower derivatives of E^{i+1}_{m+1} and E^i_m and $\alpha^u_{m+1,i+1} := [(A^{uu}_m \circ g^i_{m+1}) + (A^{su}_m \circ g^i_{m+1})(E^i_m \circ g^i_{m+1})]^{-1}$. Letting $\alpha^s_{m,i} := (A^{ss}_{m-1} \circ g^{i-1}_m) - E^i_m(A^{su}_{m-1} \circ g^{i-1}_m)$ this yields

$$\begin{aligned} D^r E^i_m &= \alpha^s_{m,i}(D^r E^{i-1}_{m-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r}\alpha^u_{m,i} + \zeta_{m,i} \\ &= \alpha^s_{m,i}(\alpha^s_{m-1,i-1} \circ g^{i-1}_m) \times \\ &\quad \times (D^r E^{i-2}_{m-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m)(Dg^{i-2}_{m-1})^{\otimes r}(\alpha^u_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r}\alpha^u_{m,i} \\ &\quad + \alpha^s_{m,i}(\zeta_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r}\alpha^u_{m,i} + \zeta_{m,i} \\ &= \dots \end{aligned}$$

Applying this inductively we obtain an expression for $D^r E^i_m$ with a leading term involving $D^r E^0_{m-i}$ between i-fold products

$$\alpha^s_{m,i}(\alpha^s_{m-1,i-1} \circ g^{i-1}_m)(\alpha^s_{m-2,i-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m) \dots$$

of terms $\alpha^s_{m-l,i-l}$ and

$$\dots (Dg^{i-3}_{m-2})^{\otimes r}(\alpha^u_{m-2,i-2} \circ g^{i-2}_{m-1} \circ g^{i-1}_m)(Dg^{i-2}_{m-1})^{\otimes r}(\alpha^u_{m-1,i-1} \circ g^{i-1}_m)(Dg^{i-1}_m)^{\otimes r}\alpha^u_{m,i}$$

of $\alpha^u_{m-l,i-l}$ and i occurrences of $(Dg^{i-l-1}_{m-l})^{\otimes r}$. This term converges to 0 uniformly as $i \to \infty$: $\|D^r E^0_{m-i}\|$ is uniformly bounded by choice of φ^0_{m-i} and $\|\alpha^s_{m-l,i-l}\|\|\alpha^u_{m-l,i-l}\| < 1$ uniformly by taking small δ. Finally, the assumption

$\mu \geq 1$ of this step ensures that the factors $(Dg_{m-l}^{i-l-1})^{\otimes r}$ cause no exponential growth.

The jth of the remaining i summands in the expression for $D^r E_m^i$ similarly consists of $\zeta_{m-j-1,i-j-1}$ between j-fold products of terms $\alpha_{m-l,i-l}^s$ and $\alpha_{m-l,i-l}^u$ as well as j occurrences of $(Dg_{m-l}^{i-l-1})^{\otimes r}$. As before, these terms will tend to 0 uniformly as $j \to \infty$ given uniform control of $\zeta_{m-j-1,i-j-1}$. These, however, involve only lower derivatives of E_l^k's which are uniformly bounded by induction assumption, as well as derivatives of order up to order r of coefficients of Df, which are bounded because $f \in C^{r+1}$. Consequently these remaining terms give partial sums of an exponentially convergent series. We already know that lower-order derivatives of E_m^i converge as $i \to \infty$ and thus conclude that the limit of E_m^i is C^r, as desired. □

Note that $(W_m^+)_p$ and $(W_m^-)_p$ for $p \in \mathbb{R}^n$ depend continuously on p: The characterization (iii) of Theorem 6.2.8 yields

Proposition 6.2.20. *If $p_l \to p \in \mathbb{R}^n$ as $l \to \infty$ and $y_l \in (W_m^+)_{p_l}$ for all $l \in \mathbb{N}$ and $y_l \to y \in \mathbb{R}^n$ as $l \to \infty$ then $y \in (W_m^+)_p$.*

Proof. Fix $L \in \mathbb{N}$. Then (ii) of Theorem 6.2.8 implies for $\nu < \mu'$ that

$$\|f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(y_l) - f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(p_l)\| \leq \nu^{-L}\|y_l - p_l\|$$

for all $l \in \mathbb{N}$. By continuity of the f_m this implies

$$\|f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(y) - f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(p)\| \leq \nu^{-L}\|y - p\|$$

and since L was arbitrary the claim follows by (iii). □

Since on any fixed compact set the assumption that y_l converges is redundant (by passing to a subsequence) this means that $(W_m^+)_{p_l} \to (W_m^+)_p$ when $p_l \to p$. Convergence here is in the pointwise sense of the proposition. Since we know that E_m^+ is continuous, we have continuity of W_m^+ together with its tangent spaces. A similar statement holds for W_m^-.

Another pertinent remark is that we obtain in fact continuous dependence of W^+ and W^- on the family f_m of maps we consider. Since the main ingredient of the proof of the Hadamard–Perron Theorem 6.2.8 was obtaining the invariant manifolds and their tangent distributions as fixed points of a contraction operator associated with the family f_m, we may use Proposition 1.1.5 to infer that the invariant manifolds depend continuously on the diffeomorphisms with respect to the C^1 topology.

Proposition 6.2.21. *The invariant manifolds (with the C^1 topology) obtained in the Hadamard–Perron Theorem 6.2.8 depend continuously on the family f_m if we use the C^1 topology defined by calling $\{f_m\}_{m\in\mathbb{N}}, \{g_m\}_{m\in\mathbb{N}}$ C^1-close when $\sup_m d_{C^1}(f_m, g_m)$ is small.*

For the next section we note: